Fourth Edition

CHAPTER

5

MECHANICS OF MATERIALS

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Analysis and Design of Beams for Bending

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Part 1: 45 min

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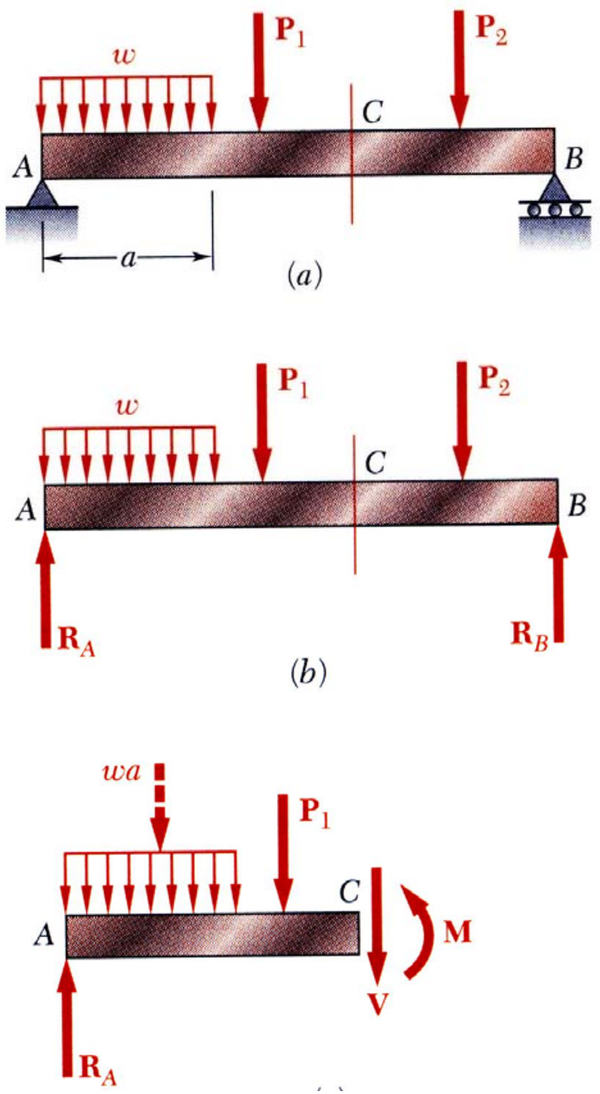
Introduction



- Objective - Analysis and design of beams
- *Beams* - structural members supporting loads at various points along the member
- Transverse loadings of beams are classified as *concentrated* loads or *distributed* loads
- Applied loads result in internal forces consisting of a *shear force* (from the shear stress distribution) and a *bending couple* (from the normal stress distribution)
- <u>*Normal stress*</u> is often the critical design criteria

$$\sigma_x = -\frac{My}{I} \qquad \sigma_m = \frac{|M|c}{I} = \frac{|M|}{S}$$

Elastic Flexure Formulas

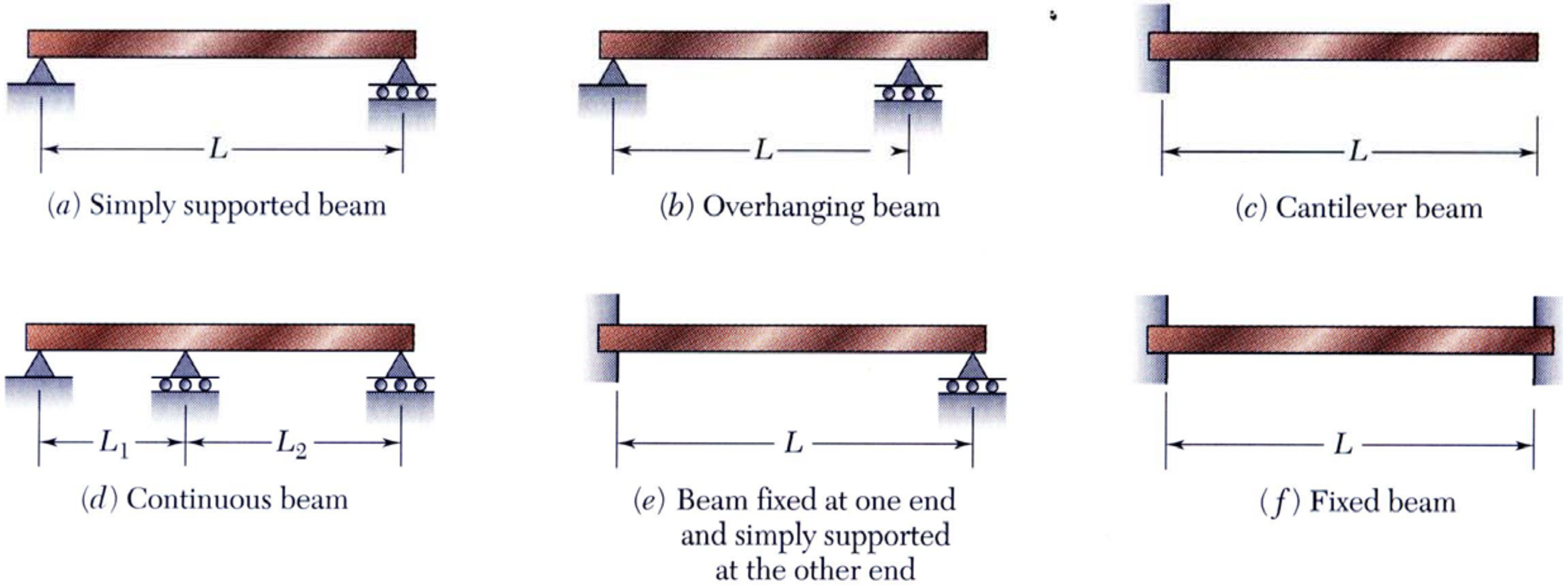
Requires determination of the location and magnitude of largest bending moment

Introduction

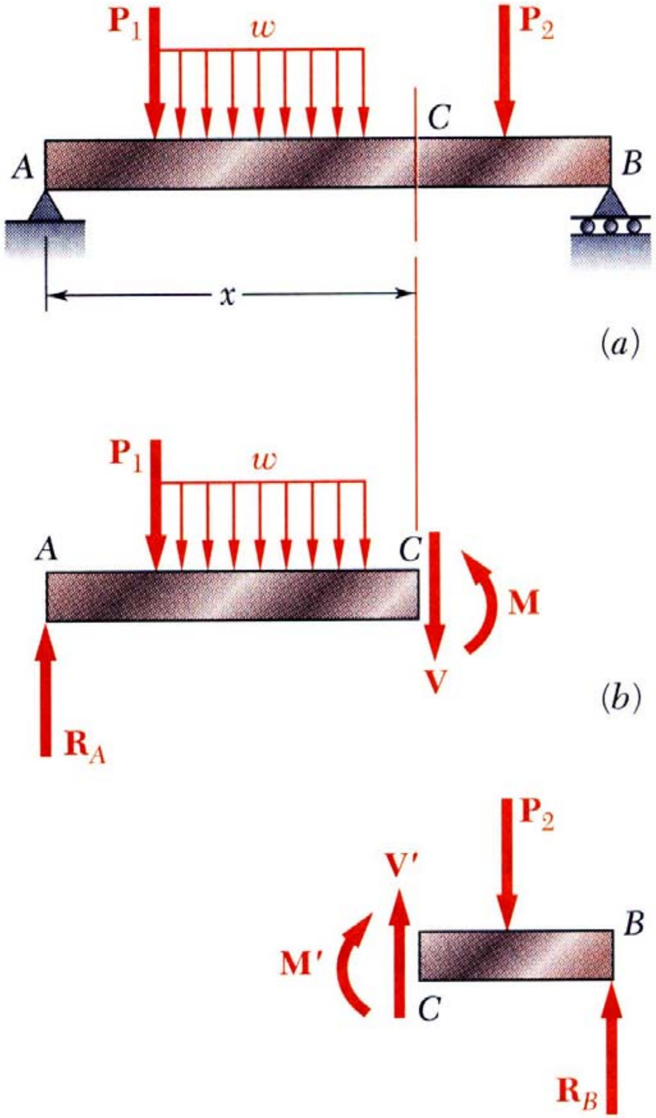
Classification of Beam Supports

Statically Determinate Beams

Statically Indeterminate Beams



Shear and Bending Moment Diagrams

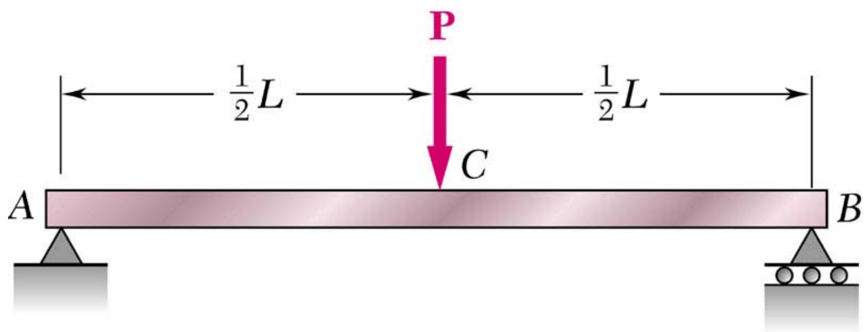


- Determination of maximum normal and shearing stresses requires identification of maximum internal shear force V and bending couple M.
- Shear force and bending couple at a point are determined by passing a section through the beam and applying an equilibrium analysis on the beam portions on either side of the section.
- Sign conventions for shear forces *V* and *V’* and bending couples *M* and *M’* -> *positive*

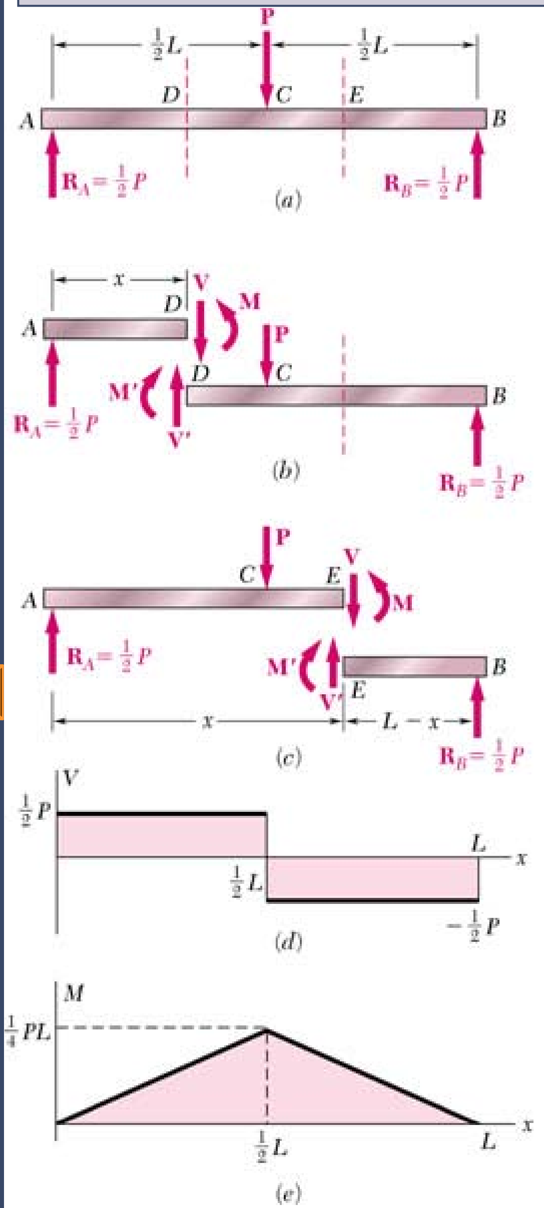
(*a*) Internal forces
(positive shear and positive bending moment)

Example 5.01

Draw the shear and bending-moment diagrams for a simply supported beam AB of span L subjected to a single concentrated load P at it midpoint C



Example 5.01



• Determination of the reactions RA=RB=1/2P

• Cut the beam at D and plot free body diagrams with positive V and M. Equilibrium equations:

$$\uparrow \sum F_y = 0;\ R_A - V = 0;\ V = R_A = 1/2P$$

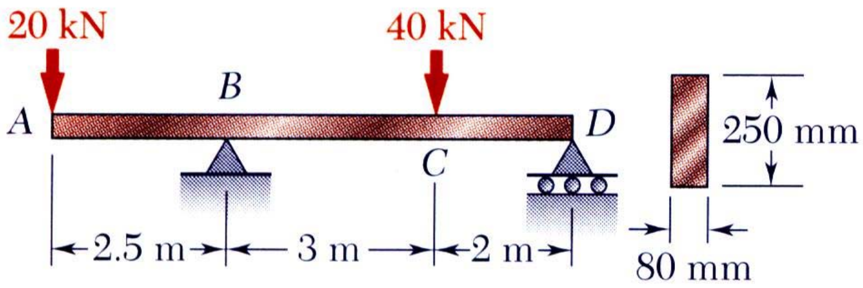
$$\circlearrowleft \sum M_D = 0;\ -R_A x + M = 0;\ M = R_A x = Px/2$$

$$\uparrow \sum F_y = 0;\ R_B + V = 0;\ V = -R_B = -1/2P$$

$$\circlearrowleft \sum M_E = R_B(L-x) - M = 0;\ M = P(L-x)/2$$

V is constant between concentrated loads and M varies linearly

Sample Problem 5.1

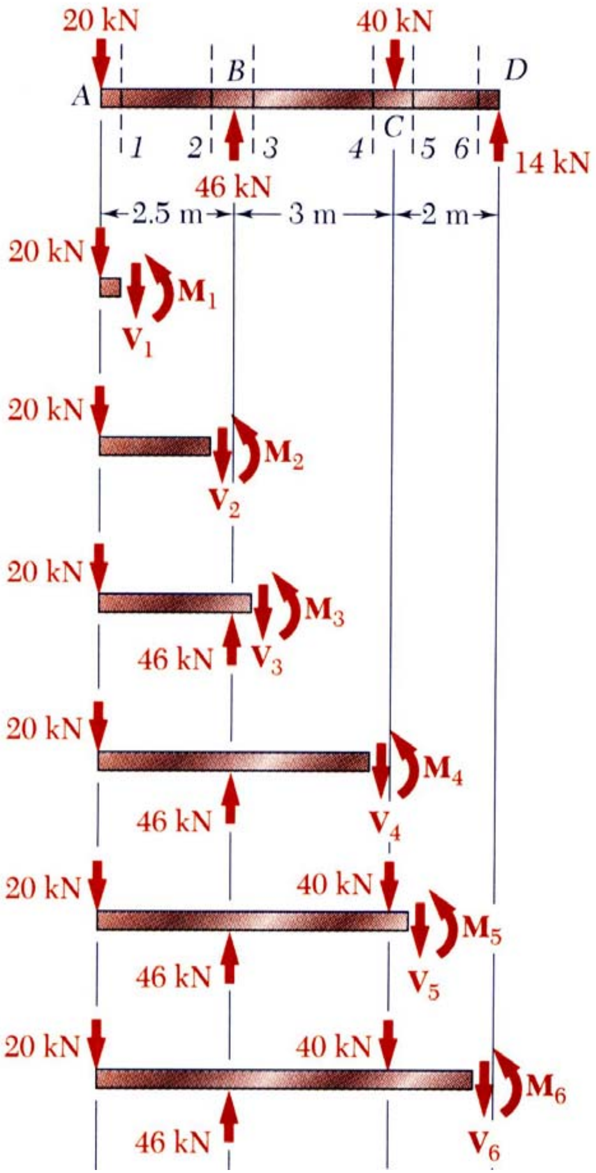


For the timber beam and loading shown, draw the shear and bend-moment diagrams and determine the maximum normal stress due to bending.

SOLUTION:

- Treating the entire beam as a rigid body, determine the reaction forces
- Section the beam at points near supports and load application points. Apply equilibrium analyses on resulting free-bodies to determine internal shear forces and bending couples
- Identify the maximum shear and bending-moment from plots of their distributions.
- Apply the elastic flexure formulas to determine the corresponding maximum normal stress.

Sample Problem 5.1



SOLUTION:

- Treating the entire beam as a rigid body

$$\sum Fy = 0 = -20 + R_B - 40 + R_D \Rightarrow R_D = 60 - R_B$$

$$\sum M_B = 0 = 20 \times 2,5 - 40 \times 3 + R_D \times 5 \Rightarrow R_D = 14$$

$$R_D = 14; R_B = 46$$

- Section the beam and apply equilibrium analyses on resulting free-bodies

$$\begin{cases} \sum F_y = 0 & -20\text{ kN} - V_1 = 0 & V_1 = -20\text{ kN} \\ \sum M_1 = 0 & (20\text{ kN})(0\text{ m}) + M_1 = 0 & M_1 = 0 \end{cases}$$

$$\begin{cases} \sum F_y = 0 & -20\text{ kN} - V_2 = 0 & V_2 = -20\text{ kN} \\ \sum M_2 = 0 & (20\text{ kN})(2.5\text{ m}) + M_2 = 0 & M_2 = -50\text{ kN}\cdot\text{m} \end{cases}$$

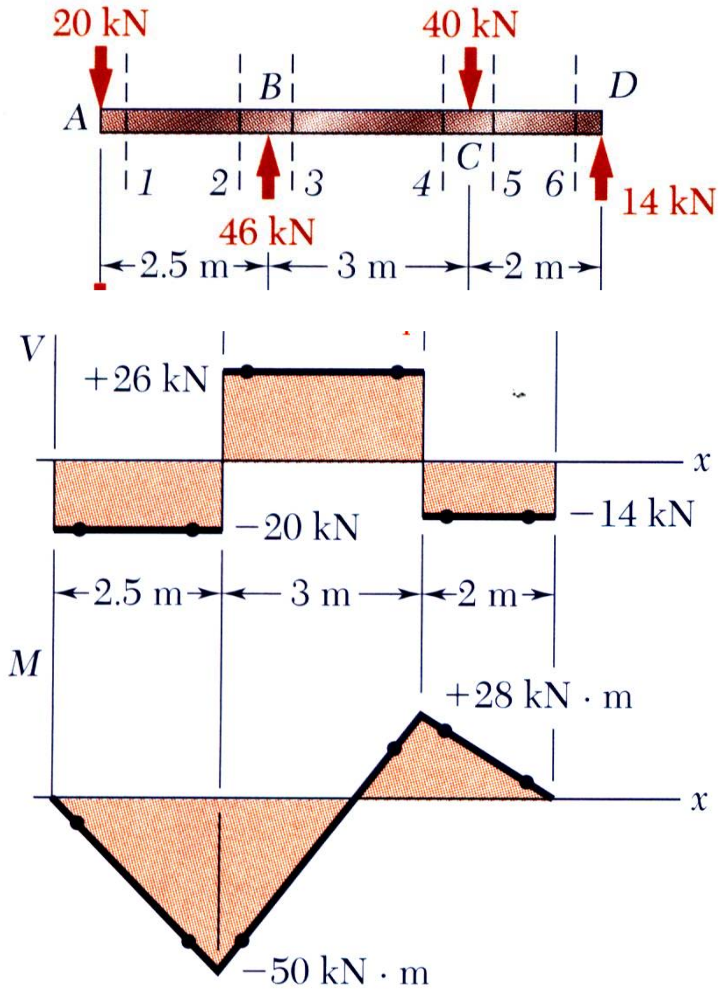
$$V_3 = +26\text{ kN} \quad M_3 = -50\text{ kN}\cdot\text{m}$$

$$V_4 = +26\text{ kN} \quad M_4 = +28\text{ kN}\cdot\text{m}$$

$$V_5 = -14\text{ kN} \quad M_5 = +28\text{ kN}\cdot\text{m}$$

$$V_6 = -14\text{ kN} \quad M_6 = 0$$

Sample Problem 5.1



V_1=-20, M_1=0; V_2=-20, M_2=-50; V_3= 26, M_3=-50; V_4=26, M_4= 28; V_5=-14, M_5=28, V_6=-14, M_6=0

- Identify the maximum shear and bending-moment from plots of their distributions.

$$V_m = 26\,\text{kN} \quad M_m = |M_B| = 50\,\text{kN}\cdot\text{m}$$

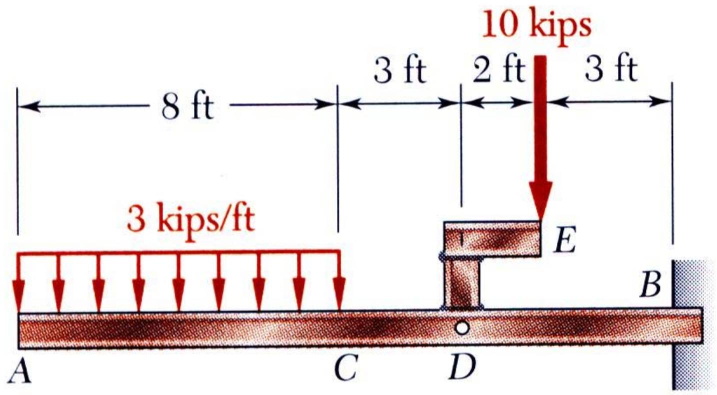
- Apply the elastic flexure formulas to determine the corresponding maximum normal stress.

$$S = \tfrac{1}{6} b h^2 = \tfrac{1}{6}(0.080\,\text{m})(0.250\,\text{m})^2$$
$$= 833.33\times10^{-6}\,\text{m}^3$$

$$\sigma_m = \frac{|M_B|}{S} = \frac{50\times10^3\,\text{N}\cdot\text{m}}{833.33\times10^{-6}\,\text{m}^3}$$

$$\boxed{\sigma_m = 60.0\times10^6\,\text{Pa}}$$

Sample Problem 5.2

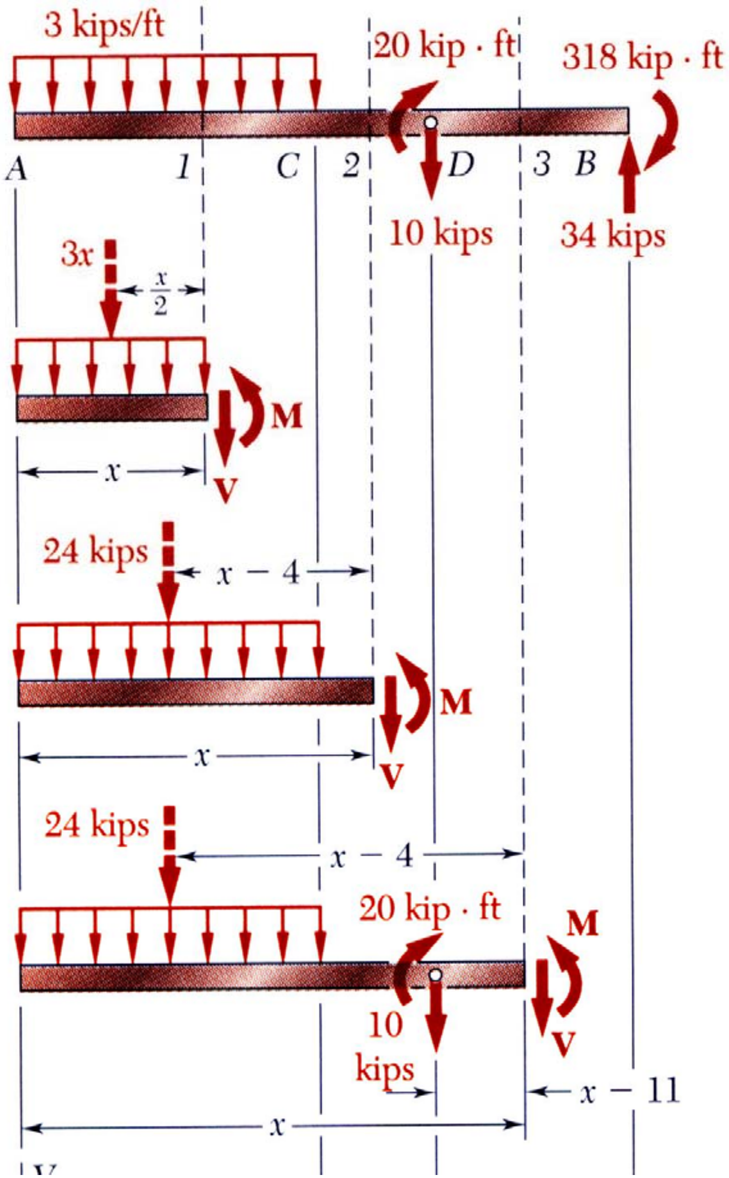


The structure shown is constructed of a W10x112 rolled-steel beam. (*a*) Draw the shear and bending-moment diagrams for the beam and the given loading. (*b*) determine normal stress in sections just to the right and left of point *D*.

SOLUTION:

- Replace the 10 kip load with an equivalent force-couple system at *D*. Find the reactions at *B* by considering the beam as a rigid body.
- Section the beam at points near the support and load application points. Apply equilibrium analyses on resulting free-bodies to determine internal shear forces and bending couples.
- Apply the elastic flexure formulas to determine the maximum normal stress to the left and right of point *D*.

Sample Problem 5.2



SOLUTION:

- Replace the 10 kip load with equivalent force-couple system at *D*. Find reactions at *B*.
- Section the beam and apply equilibrium analyses on resulting free-bodies.

From A to C :

$$\sum F_y = 0 \quad -3x - V = 0 \qquad V = -3x \text{ kips}$$

$$\sum M_1 = 0 \quad (3x)\left(\tfrac{1}{2}x\right) + M = 0 \qquad M = -1.5x^2 \text{ kip} \cdot \text{ft}$$

From C to D :

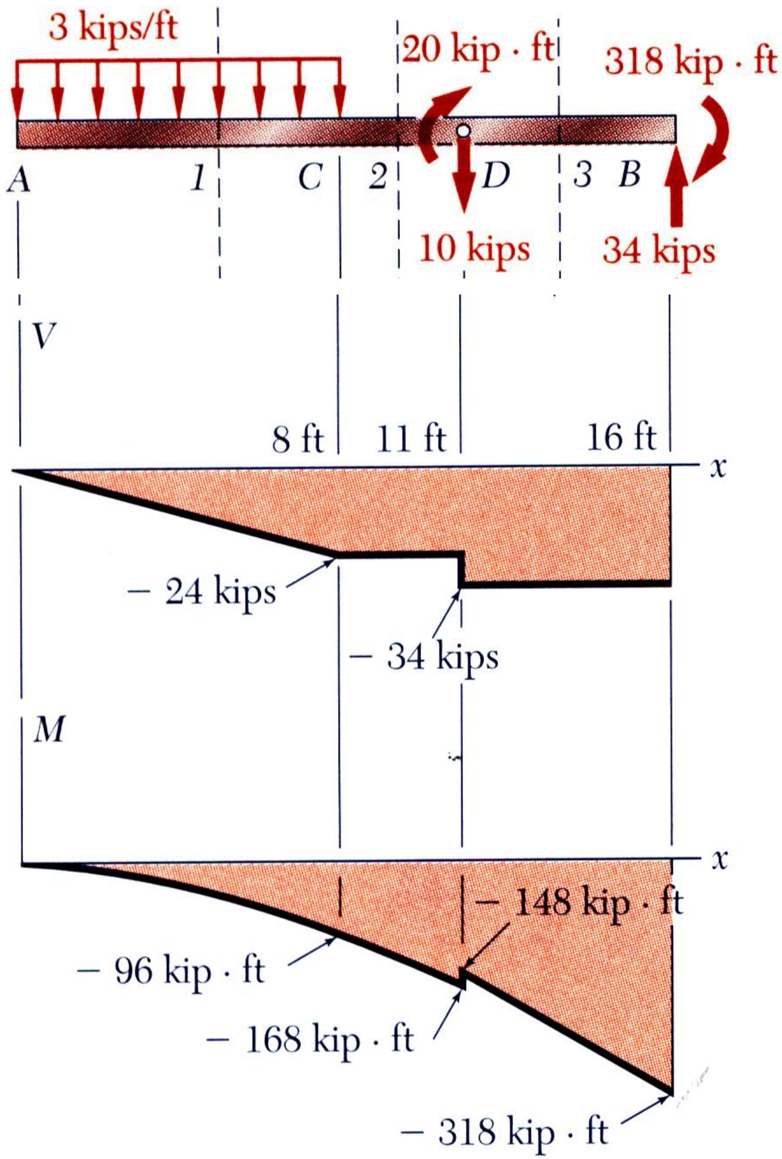
$$\sum F_y = 0 \quad -24 - V = 0 \qquad V = -24 \text{ kips}$$

$$\sum M_2 = 0 \quad 24(x-4) + M = 0 \qquad M = (96 - 24x) \text{ kip} \cdot \text{ft}$$

From D to B :

$$V = -34 \text{ kips} \qquad M = (226 - 34x) \text{ kip} \cdot \text{ft}$$

Sample Problem 5.2



- Apply the elastic flexure formulas to determine the maximum normal stress to the left and right of point *D*.

From Appendix C for a W10x112 rolled steel shape, $S = 126\ \text{in}^3$ about the *X*-*X* axis.

To the left of D:

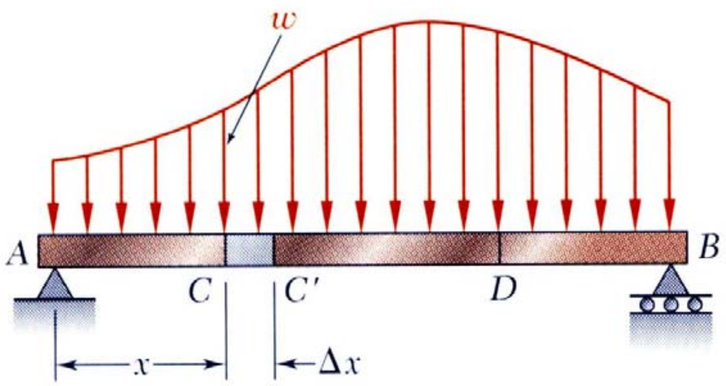
$$\sigma_m = \frac{|M|}{S} = \frac{2016\,\text{kip}\cdot\text{in}}{126\,\text{in}^3} \qquad \boxed{\sigma_m = 16.0\,\text{ksi}}$$

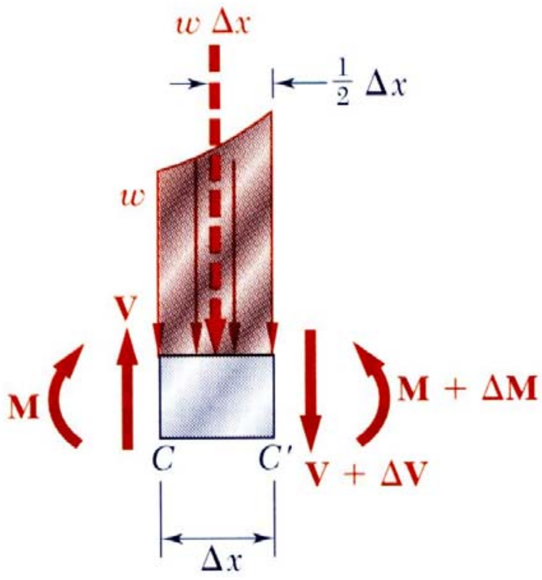
To the right of D:

$$\sigma_m = \frac{|M|}{S} = \frac{1776\,\text{kip}\cdot\text{in}}{126\,\text{in}^3} \qquad \boxed{\sigma_m = 14.1\,\text{ksi}}$$

- Concentrated loads: V constant
  M varies linearly
- Distributed load: V varies linearly
  M parabola

Relations Among Load, Shear, and Bending Moment





- Relationship between load and shear:

$$\sum F_y = 0: \quad V - (V + \Delta V) - w\,\Delta x = 0$$

$$\Delta V = -w\,\Delta x$$

$$\frac{dV}{dx} = -w$$

$$V_D - V_C = -\int_{x_C}^{x_D} w\,dx$$

- Relationship between shear and bending moment:

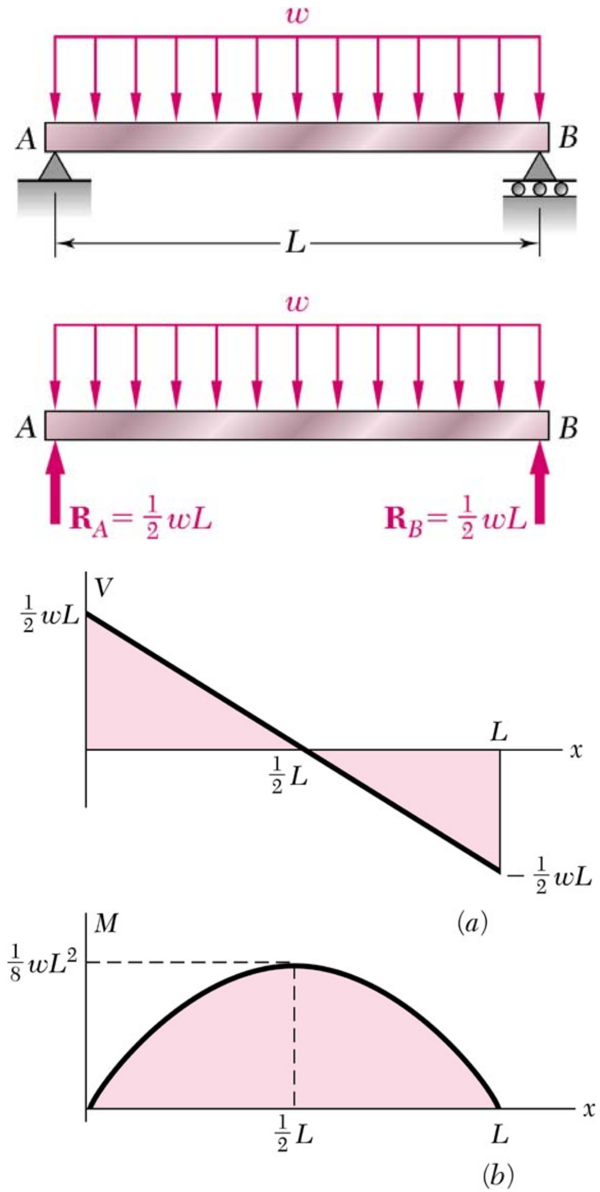
$$\sum M_{C'} = 0: \quad (M + \Delta M) - M - V\,\Delta x + w\,\Delta x \frac{\Delta x}{2} = 0$$

$$\Delta M = V\,\Delta x - \tfrac{1}{2} w (\Delta x)^2 \Rightarrow \frac{\Delta M}{\Delta x} = V - \frac{1}{2} w\,\Delta x$$

$$\Delta x \to 0; \frac{dM}{dx} = V$$

$$M_D - M_C = \int_{x_C}^{x_D} V\,dx$$

Example 5.03



Draw the shear and bending moment diagrams for the simply supported beam and determine the maximum value of the bending moment.

SOLUTION:

•$R_A = R_B = wL/2$

•Determination of V and M at any distance from A:

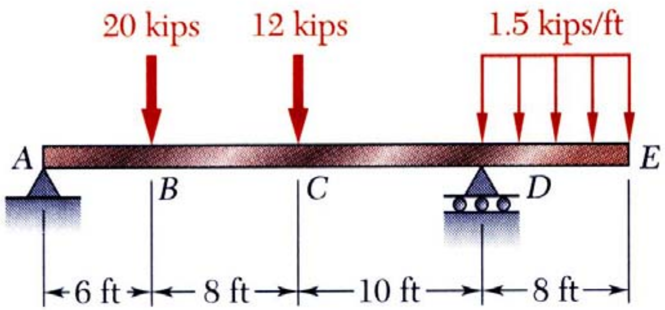
$$V - V_A = -\int_0^x w.dx = -wx$$

$$V = V_A - wx = \frac{1}{2}wL - wL = w(\frac{1}{2}L - x)$$

$$M - M_A = \int_0^x V.dx; M_A = 0$$

$$M = \int_0^x w(\frac{1}{2}L - x)dx = \frac{1}{2}w(Lx - x^2) \Rightarrow M_{max} \frac{wL^2}{8}$$

Sample Problem 5.3

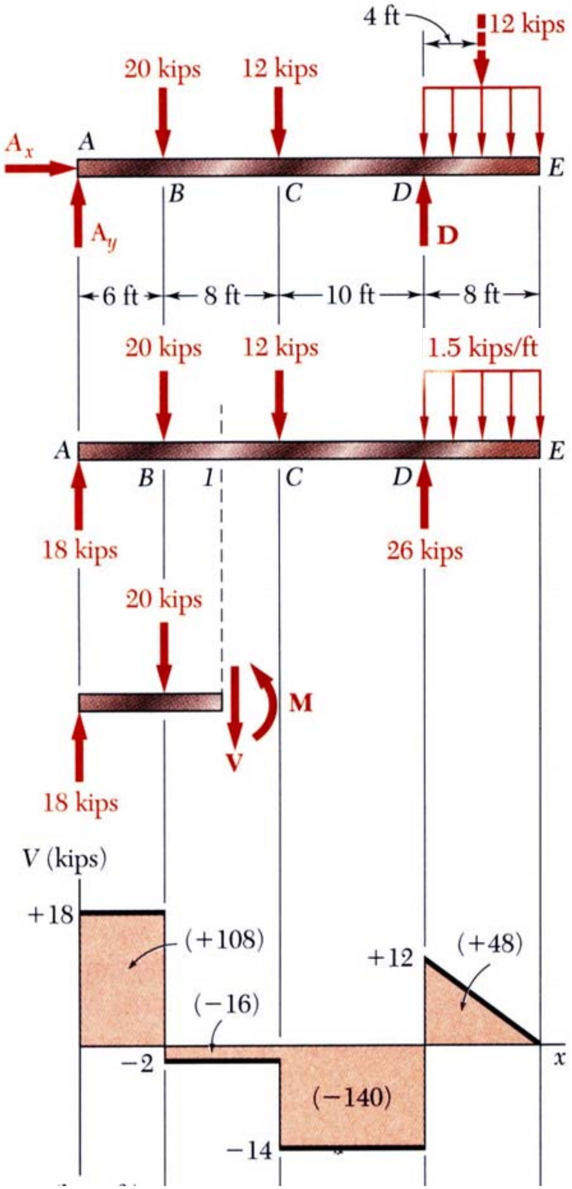


Draw the shear and bending moment diagrams for the beam and loading shown.

SOLUTION:

- Taking the entire beam as a free body, determine the reactions at *A* and *D*.
- Apply the relationship between shear and load to develop the shear diagram.
- Apply the relationship between bending moment and shear to develop the bending moment diagram.

Sample Problem 5.3



SOLUTION:

- Taking the entire beam as a free body, determine the reactions at *A* and *D*.

$$\sum M_A = 0$$

$$0 = D(24\,\text{ft}) - (20\,\text{kips})(6\,\text{ft}) - (12\,\text{kips})(14\,\text{ft}) - (12\,\text{kips})(28\,\text{ft})$$

$$D = 26\,\text{kips}$$

$$\sum F_y = 0$$

$$0 = A_y - 20\,\text{kips} - 12\,\text{kips} + 26\,\text{kips} - 12\,\text{kips}$$

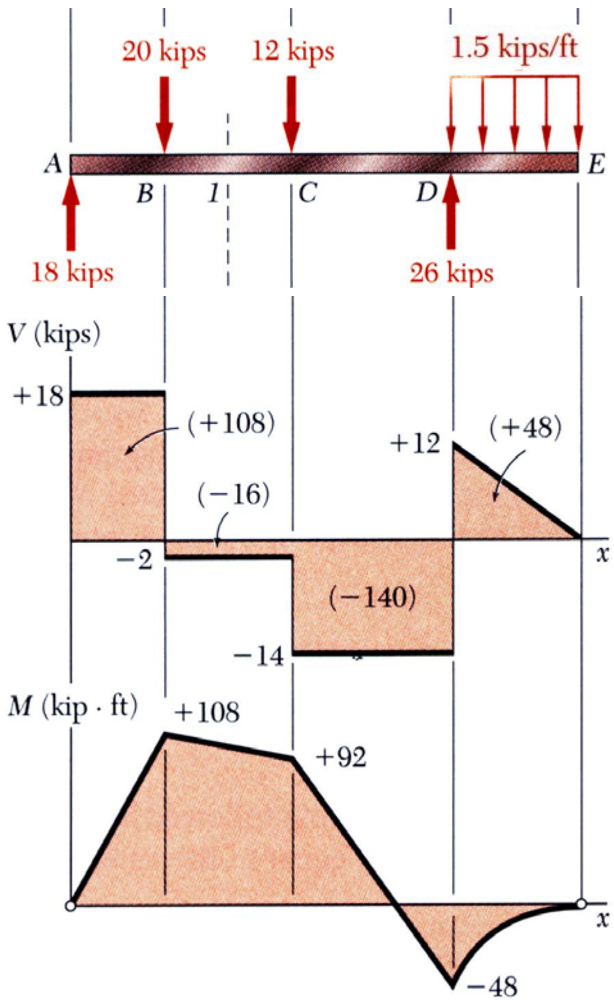
$$A_y = 18\,\text{kips}$$

- Apply the relationship between shear and load to develop the shear diagram.

$$\frac{dV}{dx} = -w \qquad dV = -w\,dx$$

- zero slope between concentrated loads
- linear variation over uniform load segment

Sample Problem 5.3



- Apply the relationship between bending moment and shear to develop the bending moment diagram.

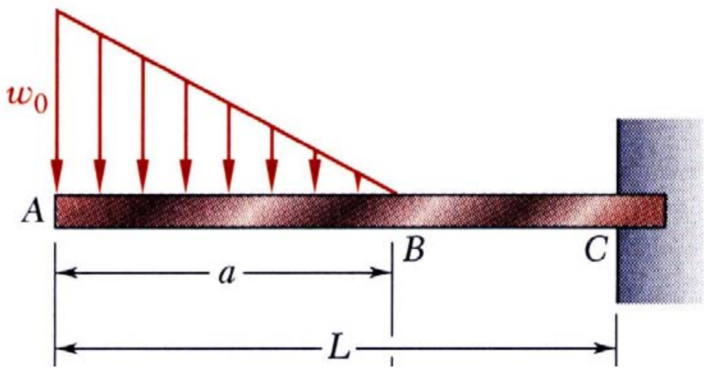
$$\frac{dM}{dx} = V \qquad dM = V\,dx$$

  - bending moment at *A* and *E* is zero
  - bending moment variation between *A*, *B*, *C* and *D* is linear
  - bending moment variation between D and *E* is quadratic
  - net change in bending moment is equal to areas under shear distribution segments

$$M_D - M_C = \int_C^D V.dx$$

  - total of all bending moment changes across the beam should be zero (108-16-140+48=0)

Sample Problem 5.5

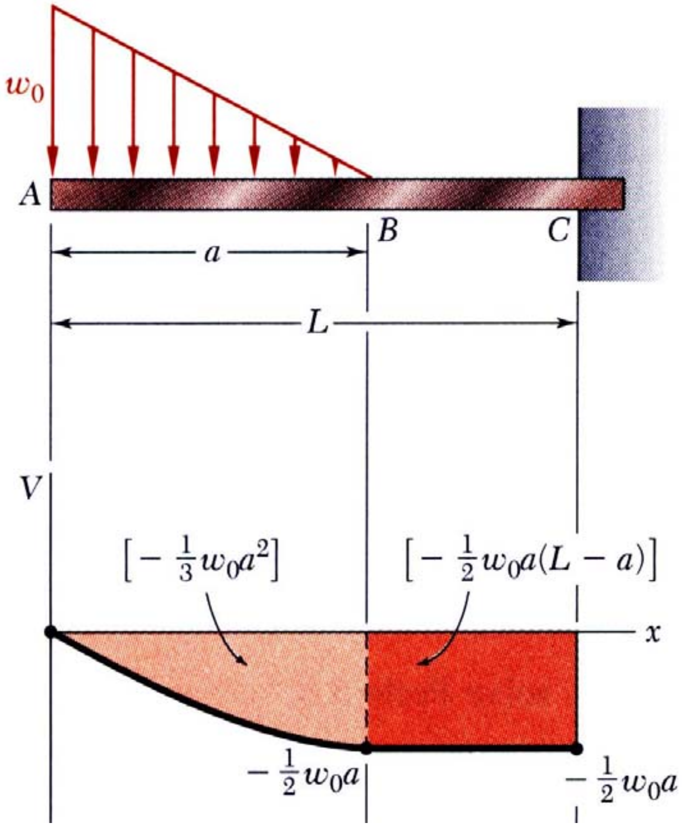


Draw the shear and bending moment diagrams for the beam and loading shown.

SOLUTION:

- Taking the entire beam as a free body, determine the reactions at *C*.
- Apply the relationship between shear and load to develop the shear diagram.
- Apply the relationship between bending moment and shear to develop the bending moment diagram.

Sample Problem 5.5



SOLUTION:

- Taking the entire beam as a free body, determine the reactions at *C*.

$$\sum F_y = 0 = -\tfrac{1}{2}w_0a + R_C \qquad R_C = \tfrac{1}{2}w_0a$$

$$\sum M_C = 0 = \tfrac{1}{2}w_0a\left(L-\frac{a}{3}\right) + M_C \qquad M_C = -\tfrac{1}{2}w_0a\left(L-\frac{a}{3}\right)$$

Results from integration of the load and shear distributions should be equivalent.

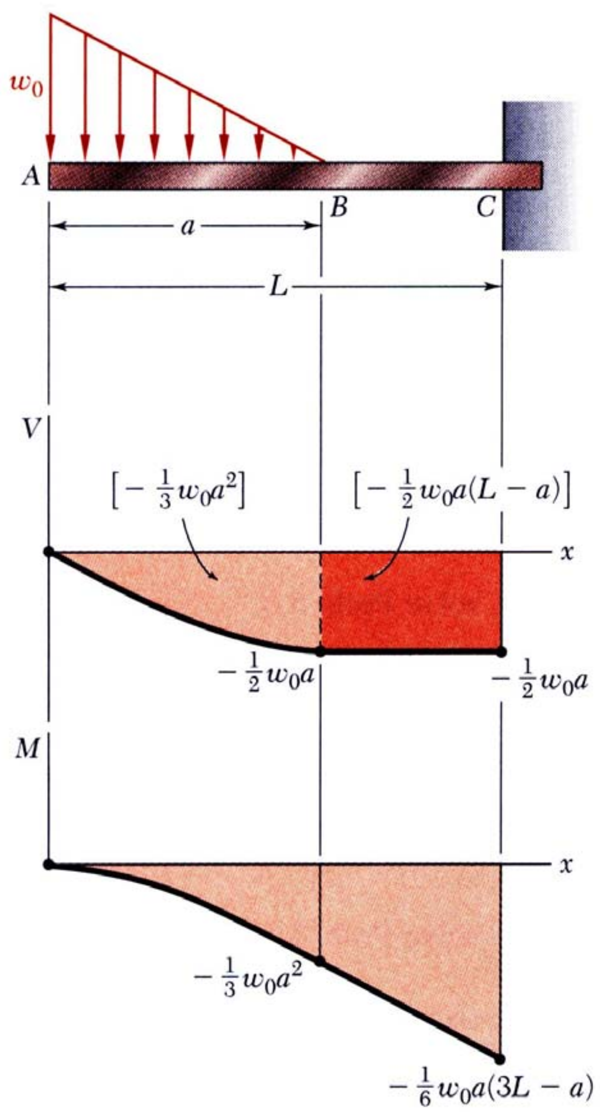
- Apply the relationship between shear and load to develop the shear diagram.

$$V_B - V_A = -\int_0^a w_0\left(1-\frac{x}{a}\right)dx = -\left[w_0\left(x-\frac{x^2}{2a}\right)\right]_0^a$$

$$V_B = -\tfrac{1}{2}w_0a = -(\mathit{area\ under\ load\ curve})$$

- No change in shear between *B* and *C*.
- Compatible with free body analysis

Sample Problem 5.5



- Apply the relationship between bending moment and shear to develop the bending moment diagram.

$$M_B - M_A = \int_0^a \left(-w_0\left(x - \frac{x^2}{2a}\right)\right)dx = \left[-w_0\left(\frac{x^2}{2} - \frac{x^3}{6a}\right)\right]_0^a$$

$$M_B = -\tfrac{1}{3}w_0a^2$$

$$M_B - M_C = \int_a^L \left(-\tfrac{1}{2}w_0a\right)dx = -\tfrac{1}{2}w_0a(L-a)$$

$$M_C = -\tfrac{1}{6}w_0a(3L-a) = \frac{a\,w_0}{2}\left(L - \frac{a}{3}\right)$$

Results at C are compatible with free-body analysis

Design of Prismatic Beams for Bending

- The largest normal stress is found at the surface where the maximum bending moment occurs.

$$\sigma_m = \frac{|M|_{\max} c}{I} = \frac{|M|_{\max}}{S}$$

- A safe design requires that the maximum normal stress be less than the allowable stress for the material used. This criteria leads to the determination of the minimum acceptable section modulus.

$$\sigma_m \leq \sigma_{all}$$

$$S_{\min} = \frac{|M|_{\max}}{\sigma_{all}}$$

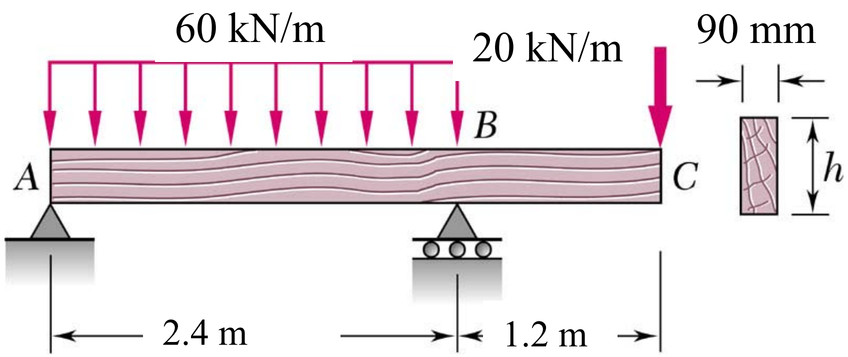
- Among beam section choices which have an acceptable section modulus, the one with the *smallest weight per unit length or cross sectional area* will be the least expensive and the best choice.

Design of Prismatic Beams for Bending

- Determine σ_{all} (If σ_{all} is the same for tension and compresion then follow 1,2,3 – Otherwise consider 4 in step 2)
- 1-Draw shear and bending-moment diagrams and determine $|M|_{max}$
- 2-Determine S_{min}
- 3-Find the dimentions of the beam to use: *b*, *h* for $S>S_{min}$

- 4-Select the beam section so that $\sigma_m <= \sigma_{all}$ for tensile and compressive stresses.

Sample Problem 5.7

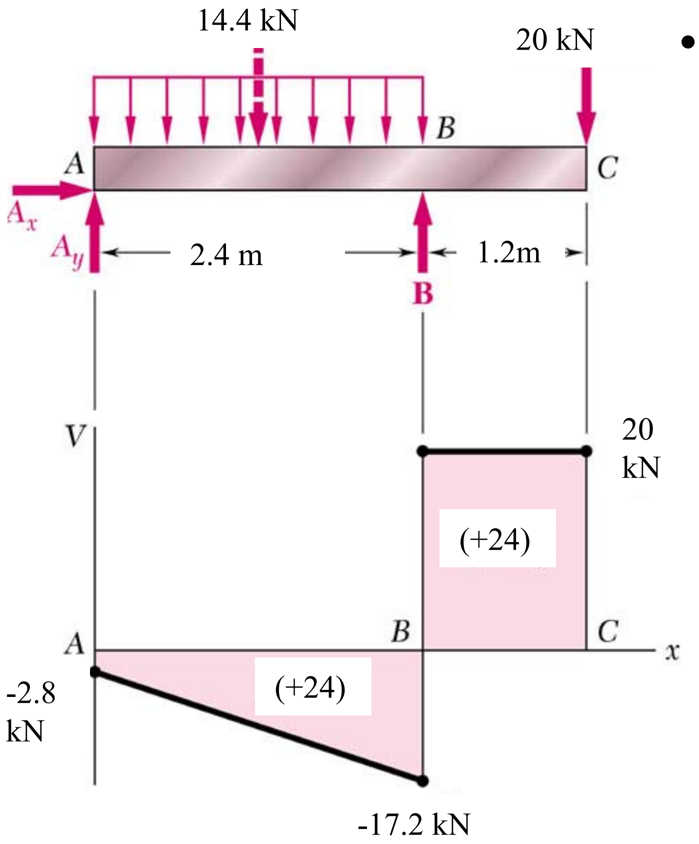


A 3.6 m-long overhanging timber beam AC is to be designed to support the distributed and concentrated loads shown. Knowing that timber of 100-mm nominal width (90-mm actual width) with a 12 MPa allowable stress is to be used, determine the minimum required depth h of the beam.

SOLUTION:

- Considering the entire beam as a free-body, determine the reactions at *A* and *B*.
- Develop the shear diagram for the beam and load distribution. From the diagram, determine the maximum bending moment.
- Determine the minimum acceptable beam section modulus. Find the value for the h.

Sample Problem 5.7



- Considering the entire beam as a free-body

$$\sum M_A = 0 = B(2.4\,\mathrm{m}) - (14.4\,\mathrm{kN})(1.2\,\mathrm{m}) - (20\,\mathrm{kN})(3.6\,\mathrm{m})$$

$$B = 37.2\,\mathrm{kN} = \mathrm{Vc}$$

$$\sum F_y = 0 = A_y + 37.2\mathrm{kN} - 14.4\,\mathrm{kN} - 20\,\mathrm{kN}$$

$$A_y = -2.8\,\mathrm{kN} = \mathrm{V_A}$$

- Plot shear diagram and determine Mmax.

$$V_B - V_A = -(area\ under\ load\ curve) = -(6kN/m)(2.4m) = -14.4kN$$

$$V_B = V_A - 14 - 4 = -2.8kN - 14.4kN = -17.2kN$$

$$V_B = -17.2\,\mathrm{kN}$$

- MA=Mc=0, Between A and B, M decreases an amount equal to area VAB, and between B and C in increases the same amount. Thus the |M|max= 24 kN.m

$$|S|_{\min} = \frac{|M|_{\max}}{\sigma_{all}} = \frac{24kN.m}{12MPa} = 2\times10^6\,mm^3$$

$$\frac{1}{6}bh^2 \ge S_{\min} \Rightarrow \frac{1}{6}(90mm)h^2 \ge 2\times10^6\,mm^3 \Rightarrow h \ge 365.2mm$$